Influence of particle inertia and Basset force on tracer dynamics: Analytic results in the small-inertia limit

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The motion of small spherical particles in three-dimensional flows is studied analytically in the weak-inertia limit. We obtain analytical results on the motion of heavy and light particles and the relative importance of the inertia and the Basset force contribution throughout the vortex region for the *ABC* flow and Hill's vortex. We find that in certain circumstances chaos is suppressed for light or heavy particles advected by the flow. We also find that the Basset force has only a weak effect in the region of a vortex center and gradually becomes more important near a separatrix; its principal effect is to change the magnitude of the drift of particles towards either a vortex center or a separatrix. [S1063-651X(97)06202-8]

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I. INTRODUCTION

The dynamical behavior of particles that are being carried away by a fluid has recently received a great deal of attention. It has been made clear that even in very simple velocity fields the motion of a passive tracer can become very complicated and even chaotic [1]. However, in many problems of interest, in a number of applications ranging from contaminant dispersion in the atmosphere to chemical engineering, one is interested in the motion of particles of a different density from that of the fluid in given flow fields. In such cases the equations of motion of the particles are considerably different from the equations of motion of the fluid parcels, as inertial and buoyancy effects make the particles have a different velocity from the ambient fluid velocity. As a consequence, the dynamical behavior of such particles can be considerably different from that of the fluid parcels (see, e.g., [2]).

In this paper we study the motion of small spherical particles in three-dimensional flows. Our aim is to give some analytical results on the motion of such particles using Melnikov-like functions.

II. MODEL

Following Druzhinin and Ostrovsky [3] (see also [4]), the velocity of a small spherical particle v, with small inertia and for small enough times, is given by

$$v=u(r,t)+\gamma\left(\frac{Du}{Dt}-\epsilon\int_{-\infty}^{t}\frac{D}{Dt'}\frac{Du}{Dt'}\frac{dt'}{\sqrt{t-t'}}\right), \qquad (2.1)$$

where the factor γ, which is a measure of the inertial effects, is given by

$$\gamma=\frac{2a^2}{9\nu}\frac{\rho_f-\rho_p}{\rho_f} \qquad (2.2)$$

and

$$\epsilon=\left(\frac{a^2}{\nu\pi}\right)^{1/2}, \qquad (2.3)$$

where ν is the viscosity, ρ_f is the fluid density, ρ_p is the density of the particles, a is the radius of the particle, D/Dt is the convective derivative, and $\mathbf{u}$ is the velocity of the ambient fluid. The first term in the large parentheses is caused by inertial forces, while the second term is the so-called Basset force and is a nonlocal in time term. The equation of motion for the particle is given by

$$\dot{\mathbf{r}}=\mathbf{v}, \qquad (2.4)$$

where the overdot denotes differentiation with respect to time. It is clearly seen from this relation that the particles, due to the effects of inertia, will deviate from the Lagrangian trajectories. The derivation of this relation is given in detail in [3] and we do not reproduce it here. It is important to note that the relation for the particle velocity is derived using perturbation methods and only the first two terms in the series solution are kept in Eq. (2.1). Usually the first inertial term is the more important. However, according to the numerical results of Druzhinin and Ostrovsky, the second term, which is the Basset force term, may become important during the processes of separatrix crossings. Also, according to these authors, heavier particles drift away from the vortex, while lighter particles are attracted by the vortex. Furthermore, the Basset force is found to reduce the drift of both heavy and light particles while regular motion (nonchaotic) was observed. Our aim in this paper is to explain these results quantitatively using perturbation theory.

We shall begin by collecting a few general results on the motion of inertial particles in a subclass of three-dimensional flows and then we shall concentrate, as examples, on the motion of inertial particles in an *ABC* flow, which is a generalization of the one studied by Druzhinin and Ostrovsky [3], and in the Hill vortex.

III. GENERAL THEORY

Let us attempt to work out a general theory for the motion of small spherical particles in a three-dimensional fluid flow. We will concentrate on flows that present a continuous symmetry group such as a spherical or helical symmetry or in general a Euler flow that is known to have a symmetry group whose infinitesimal generator is the vorticity. For the motion of passive tracers of the same density as the fluid ($\gamma=0$), an ingenious construction was proposed by Mezic and Wiggins [5], which reduced the equations of motion to a two-dimensional Hamiltonian system and a third degree of freedom that was decoupled from the other two. We will show that for most of the cases considered by Mezic and Wiggins [5] the same is true for particles when inertial effects are introduced. For simplicity we neglect, for the time being, the effect of the Basset force and concentrate on the inertial force.

In a vectorial representation we can rewrite the results of Mezic and Wiggins in the following form: The equation of motion for the passive tracer in the original coordinates (x_1,x_2,x_3) is

$$\dot{\mathbf{r}}=\mathbf{u}(x_1,x_2,x_3), \tag{3.1}$$

where $\mathbf{r}=x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k}$. According to [5], there is a coordinate system $\mathbf{z}=(z_1,z_2,z_3)$ that takes the form

$$\dot{z}_1=\frac{\partial H(z_1,z_2)}{\partial z_2},$$

$$\dot{z}_2=-\frac{\partial H(z_1,z_2)}{\partial z_1}, \tag{3.2}$$

$$\dot{z}_3=k_3(z_1,z_2),$$

which means that in this new coordinate system the vector $\dot{\mathbf{r}}$ is transformed as

$$\dot{\mathbf{r}}=h_1\dot{z}_1\mathbf{e}_1+h_2\dot{z}_2\mathbf{e}_2+h_3\dot{z}_3\mathbf{e}_3 \tag{3.3}$$

and the velocity vector is transformed to

$$\mathbf{u}=h_1\frac{\partial H(z_1,z_2)}{\partial z_2}\mathbf{e}_1-h_2\frac{\partial H(z_1,z_2)}{\partial z_1}\mathbf{e}_2+h_3k_3(z_1,z_2)\mathbf{e}_3. \tag{3.4}$$

In the above relations h_1,h_2,h_3 are the structure functions of the transformation $\mathbf{x}\rightarrow\mathbf{z}$ and $\mathbf{e}_i$, $i=1,2,3$, are the unit vectors in the new coordinate system. We will assume that the new coordinate system is orthogonal.

We now rewrite the inertial force in a coordinate-free notation as

$$F=\gamma\frac{Du}{Dt}=\gamma\left[\nabla\left(\frac{1}{2}u^2\right)+\boldsymbol{\omega}\times\mathbf{u}\right]. \tag{3.5}$$

Then under the assumption of orthogonality, the new equations of motion for the particle including inertial effects are

$$\dot{z}_1=\frac{\partial H(z_1,z_2)}{\partial z_2}+F_1,$$

$$\dot{z}_2=-\frac{\partial H(z_1,z_2)}{\partial z_1}+F_2, \tag{3.6}$$

$$\dot{z}_3=k_3(z_1,z_2)+F_3,$$

where the inertial forces take the form

$$F_1=2k_3H_{,2}\frac{h_{1,3}}{h_1}-\frac{h_3h_{3,1}}{h_1^2}k_3^2-\frac{h_2h_{2,1}}{h_1^2}H_{,1}^2-2H_{,1}H_{,2}\frac{h_{1},2}{h_1}$$
$$-H_{,1}H_{,22}+H_{,2}H_{,12}+H_{,2}^2\frac{h_{1,1}}{h_1},$$

$$F_2=-H_{,2}^2\frac{h_1h_{1,2}}{h_2^2}+H_{,1}^2\frac{h_{2,2}}{h_2}+H_{,1}H_{,12}^2-\frac{h_3h_{2,2}}{h_2^2}k_3^2$$
$$-2H_{,1}H_{,2}\frac{h_{2,1}}{h_2}-H_{,2}H_{,1}^2-2k_3H_{,1}\frac{h_{2,3}}{h_2}, \tag{3.7}$$

$$F_3=-\frac{h_1h_{1,3}}{h_3^2}H_{,2}^2-\frac{h_2h_{2,3}}{h_3^2}H_{,1}^2+\frac{h_{3,3}}{h_3}k_3^2-2H_{,1}k_3\frac{h_{3,2}}{h_3}$$
$$-H_{,1}k_{3,2}+2H_{,2}k_3\frac{h_{3,1}}{h_3}+H_{,2}k_{3,1},$$

where $,i$ denotes the derivative with respect to the coordinate z_i and similar notation is used for the second derivatives.

We observe that the behavior of this perturbed system that models the effects of particle inertia will depend on the nature of the structure functions. What Mezic and Wiggins [5] show is that the Jacobian of the transformation will be independent of the coordinates and in fact will be equal to one. This imposes the constraint that $h_1h_2h_3=1$. However, apart from this there is little we can say in general. If the symmetry group that was used for the reduction of the original system to the form (3.6) corresponds to some geometrical symmetry that is imposed by the geometry of the problem such as a cylindrical, spherical, or helical symmetry, then it is easy to see that these structure functions will not depend on the third ''neglected'' coordinate z_3. As a consequence, the system of equations (3.6) decouple (as in the case of particles of the same density as the fluid) to a two-dimensional system that depends only on (z_1,z_2) and a third equation that again depends only on these variables and so can be integrated using quadrature. Furthermore, this two-dimensional system is a Hamiltonian system perturbed by a non-Hamiltonian (not necessarily dissipative) perturbation. As a consequence, we see that in such cases the effect of particle inertia cannot introduce chaotic behavior, unless the flow is perturbed in such a way as to break the original symmetry. Also using well-known results about the behavior of such two-dimensional systems we can develop perturbative tools based on the Melnikov function to study bifurcations of limit cycles from centers of the unperturbed Hamiltonian system and separatrix cycles [6]. As a sample of these techniques we give a criterion on the existence of a limit cycle for the particle dynamics: If the unperturbed system has a one-parameter family of periodic orbits

$$\Gamma_a:x=\sigma_a(t), \tag{3.8}$$

where the functions $\sigma_a(t)$ have minimum periods T_a and a belongs to an indexing set I that is either a finite or semifinite open interval in R, then we can calculate the Melnikov function along this cycle

$$M(a,\gamma)=\int_0^{T_a}\overline{\nabla}H\wedge\mathbf{F}dt, \tag{3.9}$$

where $\overline{\nabla}H=(H_{,2},-H_{,1})$, $\mathbf{F}=\gamma(F_1,F_2)$, and the wedge product $\wedge$ is defined in R^2 as $x\wedge y=x_1y_2-y_1x_2$. If there exists an $a_0\in I$ such that $M(a_0,\gamma)=0$ and this is a simple zero, then for all sufficiently small γ there is a unique hyperbolic limit cycle in an $O(\gamma)$ neighborhood of Γ_{a_0}. If this function is never zero, then there is no cycle in an $O(\gamma)$ neighborhood of the unperturbed periodic orbit. The proof of this theorem can be found in [6]. Thus, for the particular cases we consider, rigorous analytical techniques can be employed to give an understanding of the effects of inertia.

An interesting question arises about the behavior of inertial particles in a Euler flow in this weak-inertia limit. From the results of [5] it is known that for a Euler flow there is always such a symmetry group whose infinitesimal generator is the vorticity and so the motion of a passive tracer of the same density as the fluid can be reduced in the form (3.2). The situation is still unclear as to whether this result can be generalized for the dynamics of inertial particles in a Euler flow. In general we cannot say whether these structure functions will be independent of z_3 for a general Euler flow. There are certain circumstances where this is true, as would be the case of an Euler flow with a spherical, cylindrical, or helical symmetry; in general some other geometrical symmetry; or, for instance, a Euler flow not having the Beltrami property and whose vorticity vector would be in a plane and would depend only on two of the variables. As all the Euler flows known to us have such symmetries, we have not found a counterexample where the effect of particle inertia will ruin this property of decoupling of the equations of motion. A definitive answer on whether this is always true would be highly desirable. Note that in the case where the unperturbed flow is Euler, the equations of motion for an inertial particle take the simple form

$$\begin{aligned}\dot{z}_1&=H_{,2}+\gamma\frac{P_{,1}}{h_1^2},\\ \dot{z}_2&=-H_{,1}+\gamma\frac{P_{,2}}{h_2^2},\\ \dot{z}_3&=k_3+\gamma\frac{P_{,3}}{h_3^2},\end{aligned} \tag{3.10}$$

where P is the pressure of the flow in the coordinate system $\mathbf{z}$ and depends only on z_1 and z_2.

Similar statements hold for the Basset force. Because of the complicated general form the equations of motion take, we do not include them here. We just note that even with the inclusion of the Basset force the system will decouple to a two-dimensional system (which now is time dependent) and a third equation that can be integrated using quadrature. Although, in principle, the two-dimensional time-dependent system could exhibit chaos, we shall see below that in the examples studied in this paper the appropriate Melnikov function does not have zeros, thus precluding chaos. Since the two-dimensional unperturbed system is a Hamiltonian system, the phase space consists of periodic trajectories or asymptotic trajectories (separatrix cycles). We then see that in the new coordinates the first-order approximation of the Basset force would be a time periodic force of the same period as the unperturbed trajectories and generally a pulse-like infinite period force near the separatrices. Similarly, Melnikov functions can be written down for the existence of periodic orbits of the time-dependent two-dimensional system. The behavior of the particles near the separatrix is harder to study analytically since there are technical difficulties in the construction of a Poincaré map for this infinite-period perturbation. However, some progress can be made using the impulselike behavior of the Basset force near the separatrix, and this approach is currently under consideration. These claims will be clarified in the next section for a particular model.

Examples where the results of Sec. II can be used are for particles in a Taylor vortex flow, the Hill vortex flow (which is a Euler flow for which the motion of the inertial particles decouples), or the ABC flow when $C=0$. In the next section we consider the motion of inertial particles in the ABC flow when $C\equiv 0$ and also in the case where C is small.

IV. PARTICLES IN THE *ABC* FLOW

We will consider the dynamics of small particles in the ABC flow, which is defined by

$$\begin{aligned}u_1&=A\sin z+C\cos y,\\ u_2&=B\sin x+A\cos z,\\ u_3&=B\cos x+C\sin y.\end{aligned} \tag{4.1}$$

In this particular case the system with $C\equiv 0$ already is in the form of Eq. (4.1). The equations of motion for the passive tracer are

$$\begin{aligned}\dot{x}&=A\sin z+C\cos y,\\ \dot{y}&=B\sin x+A\cos z,\\ \dot{z}&=B\cos x+C\sin y.\end{aligned} \tag{4.2}$$

In this particular case the system with $C\equiv 0$ already is in the form of Eq. (3.2). The equations of motion for a small particle are

$$\begin{aligned}\dot{x}&=u_1+\gamma(AB\cos x\cos z-BC\sin x\sin y)+\gamma B_{f_1},\\ \dot{y}&=u_2+\gamma(BC\cos x\cos y-AC\sin y\sin z)+\gamma B_{f_2},\\ \dot{z}&=u_3+\gamma(AC\cos y\cos z-AB\sin x\sin z)+\gamma B_{f_3},\end{aligned} \tag{4.3}$$

where B_{f_i}, $i=1,2,3$, are the Basset force corrections for this particular flow. One can see that the case $\gamma=0$ and $C=0$ is an integrable case. We will assume that γ is of the same order of magnitude as C and neglect second-order effects in the perturbation expansion.

We will first consider the effect of particle inertia and the Basset force on the particles in the y-independent cellular flow $(C=0)$. When $C=0$ and $\gamma=0$ the system is in Hamiltonian form for x and z and there is a constant of the motion (the Hamiltonian) $h=B\sin x+A\cos z$. The solution of the problem can be written in terms of elliptic functions in the form

$$\dot{x}=k\lambda\{\mathrm{cn}[\lambda(t+t_0)]+\mathrm{cn}[\lambda(t-t_0)]\},$$

$$\dot{z}=k\lambda\{\mathrm{cn}[\lambda(t+t_0)]-\mathrm{cn}[\lambda(t-t_0)]\}, \tag{4.4}$$

where an overdot denotes differentiation with respect to time,

$$k^2=\frac{(A+B)^2-h^2}{4AB},$$

$$\lambda^2=AB, \tag{4.5}$$

and t_0 is the positive solution of the equation

$$\sin^2\left(\frac{1}{2}z_0\right)=k^2\mathrm{sn}^2(\lambda t_0|k^2) \tag{4.6}$$

[we have chosen the time origin such that $x(0)=\pi/2$ and $z(0)=z_0$]. Expanding the equations of motion in powers of γ, we can write the Melnikov function (which essentially gives us the change of the unperturbed Hamiltonian over an unperturbed period) due to the effect of particle inertia as $\underline{M}_1=\gamma M_1$ with

$$M_1=-\langle\dot{x}\ddot{z}-\dot{z}\ddot{x}\rangle, \tag{4.7}$$

where the angular brackets denote an averaging over the unperturbed period and x and z denote the unperturbed solutions. Using the Fourier expansions of the elliptic functions in terms of the nome (see [7]) we obtain

$$\dot{x}=\frac{4\pi\lambda}{K(k)}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1+q^{2n+1}}\cos\left(\frac{(2n+1)\pi\lambda t_0}{2K}\right)\times\cos\left(\frac{(2n+1)\pi\lambda t}{2K}\right),$$

$$\dot{z}=-\frac{4\pi\lambda}{K(k)}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1+q^{2n+1}}\sin\left(\frac{(2n+1)\pi\lambda t_0}{2K}\right)\times\sin\left(\frac{(2n+1)\pi\lambda t}{2K}\right). \tag{4.8}$$

We can easily perform the integrations to obtain the Melnikov function

$$M_1=\left(\frac{4\pi\lambda}{K}\right)^2\pi\sum_{n=0}^{\infty}\left(\frac{q^{n+1/2}}{1+q^{2n+1}}\right)^2(2n+1)\sin\left(\frac{(2n+1)\pi\lambda t_0}{K}\right). \tag{4.9}$$

Here $K=K(k)$ is the complete elliptic integral of first kind. We now obtain the contribution the Melnikov function due to the Basset force. From Sec. II we see that the first-order contribution to the Basset force is

$$B_{f_1}=-\int_{-\infty}^{t}\frac{d^3x(t')}{d^3t'}\frac{1}{\sqrt{t-t'}}dt', \tag{4.10}$$

$$B_{f_3}=-\int_{-\infty}^{t}\frac{d^3z(t')}{d^3t''}\frac{1}{\sqrt{t-t'}}dt' \tag{4.11}$$

where $x(t)$ and $z(t)$ are the unperturbed solutions. Using the Fourier-series expansion we can perform the integration term by term and get the result

$$B_{f_1}=\frac{4\pi\lambda}{K}\sum_{n=0}^{\infty}G_n\cos\left(\frac{(2n+1)\pi\lambda t_0}{2K}\right)\frac{(2n+1)^{3/2}\lambda^{3/2}}{4K^{3/2}}\times\left[\sin\left(\frac{(2n+1)\pi\lambda t}{2K}\right)+\cos\left(\frac{(2n+1)\pi\lambda t}{2K}\right)\right], \tag{4.12}$$

$$B_{f_3}=\frac{4\pi\lambda}{K}\sum_{n=0}^{\infty}G_n\sin\left(\frac{(2n+1)\pi\lambda t_0}{2K}\right)\frac{(2n+1)^{3/2}\lambda^{3/2}}{4K^{3/2}}\times\left[\sin\left(\frac{(2n+1)\pi\lambda t}{2K}\right)-\cos\left(\frac{(2n+1)\pi\lambda t}{2K}\right)\right], \tag{4.13}$$

where

$$G_n=\frac{q^{n+1/2}}{1+q^{2n+1}}. \tag{4.14}$$

We note that the Basset force is a periodic force with the same period as the unperturbed motion and in some sense resonates with it. Close to the separatrix the period of the force (like the period of the motion) becomes infinite. We also see that since close to the separatrix the nome takes the value $q=1$, the Fourier series for the Basset force reduces to terms of the form $\Sigma_{n=0}(2n+1)^{3/2}\cos[(2n+1)s]$ and is formally equivalent to some fractional derivative of a δ function.

The Melnikov contribution due to the Basset force is $\underline{M}_2=\gamma\epsilon M_2$ where

$$M_2=\langle\dot{x}(t)B_{f_3}(t+\tau)-\dot{z}(t)B_{f_1}(t+\tau)\rangle. \tag{4.15}$$

Here the angular brackets denote integration over t over an unperturbed period. Notice the introduction of a phase factor τ that comes from the fact that the Basset force is time dependent. However, a more careful examination of the exact form of the Basset force allows us to conclude that the only relevant value of the phase factor is 0. This is easily seen by noting that to lowest order the Basset force is an integral

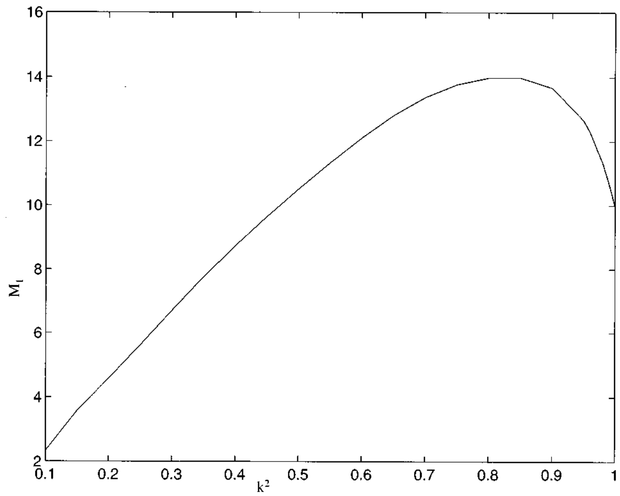

FIG. 1. Inertial contribution to the Melnikov function M_1 in the *ABC* flow with $A=2$ and $B=1$ as a function of k^2. This Melnikov function is seen to keep its sign for all values of k^2 and takes a nonzero value at the separatrix $(k=1)$.

over the unperturbed trajectory and the integration is performed over a memory kernel with time ranging from $-\infty$ to the present time. Thus, if the unperturbed orbit is taken with some phase the force will have the same phase and there will be no phase dependence in the Melnikov function. However, this might not always be the case if the memory term is different, that is, if the time integration is between a finite past time t_0 to infinity. Thus we leave the phase dependence in the Melnikov function for completeness. In the remainder of this paper, when referring to the particular form of the Basset force that ensures that the phase factor is zero, we will use $M_2(0)$. Using the Fourier series for the solutions we obtain the result for the Melnikov function

$$M_2=-\frac{8\pi^4\lambda^{5/2}}{K^{5/2}}\sum_{n=0}^{\infty} G_n^2(2n+1)^{3/2}\sin\left(\frac{(2n+1)\pi\lambda t_0}{K}\right)\times\left[\cos\left(\frac{(2n+1)\lambda\pi\tau}{2K}\right)-\sin\left(\frac{(2n+1)\pi\lambda\tau}{2K}\right)\right]. \tag{4.16}$$

We can interpret the sum of the two Melnikov functions as the change of value of the unperturbed Hamiltonian over an unperturbed period. In fact, using a multiple-time-scale perturbation technique on the equations of motion (see the Appendix) we find that

$$\dot{h}=\gamma M_1-\epsilon\gamma M_2(0), \tag{4.17}$$

where the overdot defines the ''slow'' time change of the unperturbed Hamiltonian and the right-hand side of the equation is a function of h. This is in agreement with the rigorous results on the meaning of the Melnikov function quoted at the end of Sec. II.

Neglecting the Basset force for a moment, we see that the Melnikov function is a number depending only on the unperturbed trajectory and of course the value of γ. In Fig. 1 we give M_1 as a function of k^2. The values of A and B chosen in this figure are 2 and 1, respectively. We notice that M_1 always keeps its sign, which is positive. This means that there is no possibility for the generation of a limit cycle for the inertial particle dynamics. Furthermore, interpreting the Melnikov function as the change of the value of the unperturbed Hamiltonian over an unperturbed period we can say something more about the behavior of particles in the flow: For a particle that is heavier than the fluid $(\gamma<0)$ its ''effective'' value of h tends to be reduced in time. Since the center of the vortex corresponds to higher values of h than the outer part of it ($h=3$ at the center of the vortex while $h=1$ at the separatrix), we can say that a heavy particle will tend to be ejected from the center of the vortex to the separatrix region. A more quantitative approach to this process can be obtained by fitting M_1 and $M_2(0)$ to a polynomial in h and using Eq. (4.17) to solve for the evolution of h in time. Furthermore, we find that the value of the Melnikov function at the separatrix is nonvanishing and of positive sign, so that the separatrix structure must break under the effects of particle inertia. The opposite results follow for light particles $(\gamma>0)$, which tend to concentrate in the center of the vortex. We notice that the value of M_1 on the separatrix for these values of A and B is nonzero. The value of M_1 on the separatrix is calculated using the closed form of the separatrix solution. Thus a particle can leave the cell under the action of the inertia force only. Here, there is no contradiction with the results of Druzhinin and Ostrovsky. In their paper they observed that a particle will not leave the cell under the action of the inertia force only. However, the flow they used was a specially symmetric case of the *ABC* flow $(A=B)$ for which the integrand of the Melnikov function identically vanishes on the separatrix. This is not generally the case as, for example, in the case considered here. In general, for an *ABC* flow

$$\dot{x}\ddot{z}-\dot{z}\ddot{x}\sim A\sin(x)+B\cos(x)-\sin(x)\cos(z)h. \tag{4.18}$$

In the special case where $A=B$ this is proportional to the value of the Hamiltonian and since for $A=B$ the value of the Hamiltonian on the separatrix solution is $h=0$, the integrand of the Melnikov function will vanish. However, in general cases this integrand will not vanish along the separatrix orbit. A more detailed analysis will be needed to reveal the global phase plane dynamics.

Including the effect of the Basset force, we see that the Melnikov function due to the Basset force depends on a phase and can take positive and negative values depending on the value of this phase. In the same spirit as before we interpret this result as that the effect of the Basset force will be ''on average'' to change the drift of particles towards either the center of the vortex or the separatrix. Looking at the minimum value of the total Melnikov function $M_1+\epsilon M_2$ over all the possible phases, we see that this minimum is clearly above zero for all values of k^2 for a range of values of ϵ. This rules out the existence of periodic orbits for the inertial particles under the influence of the Basset force for small enough γ (Fig. 2). Thus the Basset force will not change the dynamics apart from changing the magnitude of the drift of particles towards the center of the vortex or the separatrix. Note that the only case where the minimum of the total Melnikov function will be negative is when ϵ is as large

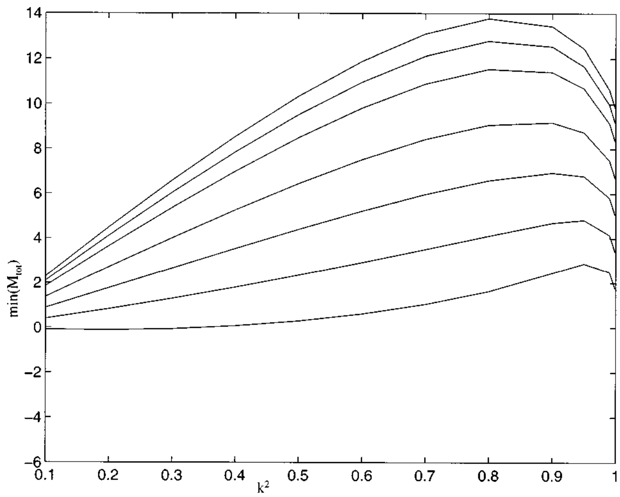

FIG. 2. Minimum over the phase of the total Melnikov function ($M_{tot}=M_1+\epsilon M_2$: inertial and Basset force contribution) as a function of k^2 for the ABC flow for different values of the parameter ϵ ($\epsilon=0.01,0.05,0.1,0.2,0.3,0.4,0.5$). ϵ quantifies the relative importance of the Basset force with respect to inertial forces. The minimum of the total Melnikov function is seen to be bounded away from zero for all values of k^2 for a large range of values of ϵ.

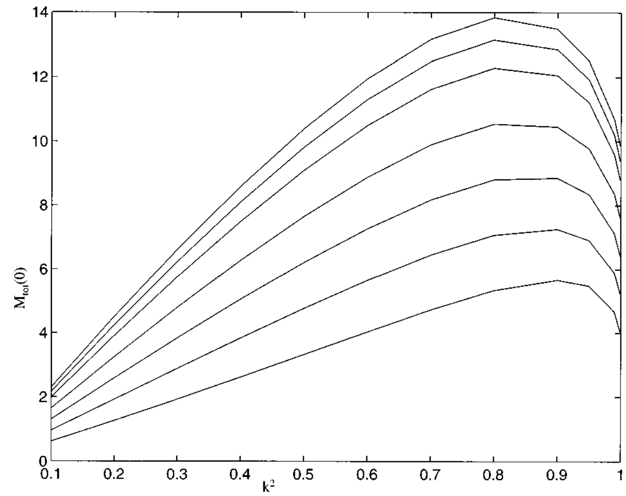

FIG. 3. Value of the total Melnikov function at phase 0, $M_{tot}(0)=M_1+\epsilon M_2(0)$, as a function of k^2 for different values of ϵ (as in Fig. 2).

as 0.5. This corresponds to large values of γ and for such values the basic equation (2.1) may not be applicable. (From the definition of ϵ and γ we see that their values are not independent.) In Fig. 3 we show the total Melnikov function $M_1+\epsilon M_2(0)$, which, as pointed out before, would correspond to the change of unperturbed Hamiltonian over a period for the special form of the Basset force.

Furthermore, we can study the relative importance of the inertia to the Basset force as we move throughout the flow using these Melnikov functions. In Fig. 4(a) we plot M_{av}, the square average over the phase of M_2:

$$M_{av}=\frac{1}{2\pi}\int_0^{2\pi} M_2(\tau)^2 d\tau,$$

and in Fig. 4(b) the ratio M_{av}/M_1. It is seen that the relative importance of the Basset force is very small for small values of k^2 (the center of the vortex) and gradually becomes more important as we move towards the separatrix. The curve has a local minimum somewhere near the separatrix (whose value is still much larger than the value at $k=0$) and shoots up near $k=1$. We believe that M_{av} (the square average of M_2 over the phase) gives us an indication of the transport of phase space area and thus believe that it is a reasonable measure of the effect of the Basset force. For the sake of comparison in Fig. 5 we plot the ratio of the maximum over the phase of M_2 to M_1 and the ratio of the value of M_2 when the phase is 0 to M_1 as functions of k^2. The first quantity slowly decreases from the center of the vortex to the separatrix and has an abrupt peak at $k=1$, while the second shows the same behavior as the square average over the phase.

Returning to the comments we made earlier we consider $M_2(0)/M_1$ as the proper measure of the importance of the Basset force compared with the inertial force. This measure clearly shows the significance of the Basset force as we move close to the separatrix region.

Summarizing, we see that using the Melnikov functions M_1 and M_2, we can reproduce the numerical results of [3] for a very similar cellular flow, namely, that heavy particles go towards the separatrix and light particles go towards the center of the vortex, and that the Basset force can influence this drift and becomes most important in the separatrix region. Thus it has to be taken into account for a correct modeling of the particle motion. An interesting point is that in general, even without the Basset force, a heavy particle can be ejected from the vortex and cross the unperturbed separatrix. Numerical integration of the system that we have performed seems to support this view. Therefore, it seems that the conclusion of [3] that separatrix crossing can only occur as a result of the Basset force is limited to particular cellular flows such as the one they studied.

We now consider the effect of the breakup of the symmetry with respect to translations in y by introducing a small value of C. We can divide the equations of motion for x and z by the equation of motion for y and consider the variable y as the new ''time.'' In the limit where γ is of the same order of magnitude as C we obtain

$$\frac{dx}{dy}=\frac{A\sin(z)}{h}+\frac{C\cos(y)}{h}+\gamma\frac{AB\cos(x)\cos(z)}{h}+B_1,$$

$$\frac{dz}{dy}=\frac{B\cos(x)}{h}+\frac{C\sin(y)}{h}-\gamma\frac{AB\sin(x)\sin(z)}{h}+B_3, \tag{4.19}$$

where B_1 and B_3 are the x and z components of the Basset force parametrized in y instead of time. It is easy to see that the Melnikov contributions due to the inertia and the Basset force will be of the same form as given earlier, but now instead of t we have to introduce y/h. There is also a Melnikov contribution due to the perturbation caused by having $C\neq 0$. This is $\bar{M}_3=C M_3$, where

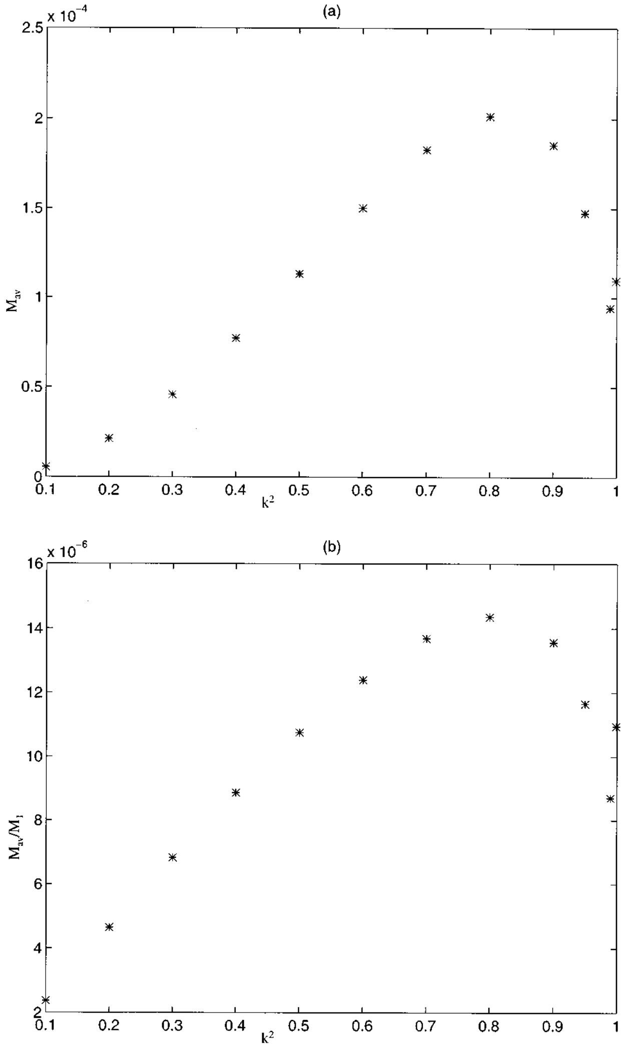

FIG. 4. (a) $M_{\rm av}$ as a function of k^2 [the square average over the phase of M_2: $M_{\rm av}=(1/2\pi)\int_0^{2\pi}M_2^2(\tau)d\tau$] and (b) $M_{\rm av}/M_1$ as a function of k^2.

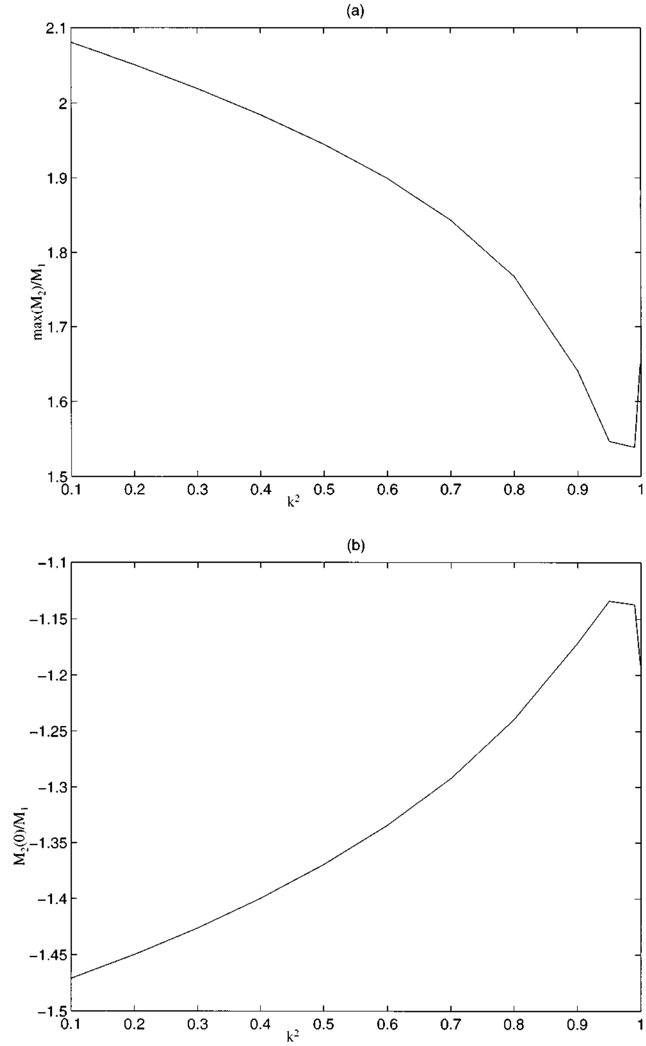

FIG. 5. (a) Ratio of the maximum of the Basset force contribution to the Melnikov function $\max_{\tau\in[0,2\pi]}[M_2(\tau)]$ to the inertial contribution M_1 and (b) ratio of the Basset force contribution calculated at zero phase $M_2(0)$ to the inertial contribution M_1 as functions of k^2.

$$M_3=\langle\dot{x}\sin(\tau+y)-\dot{z}\cos(\tau+y)\rangle. \tag{4.20}$$

Here the angular brackets denote averaging over a period (remember that now the equations have as independent variable y instead of t, and this introduces a minor modification in the period of the motion). Notice also that now a part of the perturbation depends in a periodic manner on the independent variable and this makes it necessary to introduce the phase factor τ. The Melnikov function M_3 can be obtained in closed form as was done in [8] and may be written as

$$M_3=2|h|\pi\,\mathrm{sech}\left(\frac{-K'h}{\lambda}\right)\sin\left(\frac{2u_0h}{\lambda}\right)\sin(\tau), \tag{4.21}$$

where

$$\mathrm{dn}^2(u_0)=\frac{A-B+h}{2A} \tag{4.22}$$

or as a Fourier series

$$M_3(\tau)=32\pi\lambda\sum_{n=0}^{\infty}\left[G_n\frac{2hK\cos(a)-(\pi\lambda+2\pi\lambda n)\sin(a)}{4h^2K^2-(2\pi\lambda n+\pi\lambda)^2}\right]$$
$$\times\sin\left(\frac{2hK}{\lambda}\right)\sin(\tau),$$
$$a=\frac{(2n+1)\pi\lambda t_0}{2K}. \tag{4.23}$$

We can now define the total Melnikov function as $M_t=CM_3+\gamma M_1+\epsilon\gamma M_2$. By studying its zeros as a function of the phase τ, we can study the generation of periodic orbits under the influence of particle inertia, the effect of introducing the y dependence of the velocity field, and the effect of the Basset force. It is easily seen that for certain values of γ this function might be bounded away from zero, thus precluding the existence of periodic orbits of the same period as the unperturbed one. Such an effect is illustrated in Fig. 6 for values $A=2$ and $B=1$. In this figure we plot the

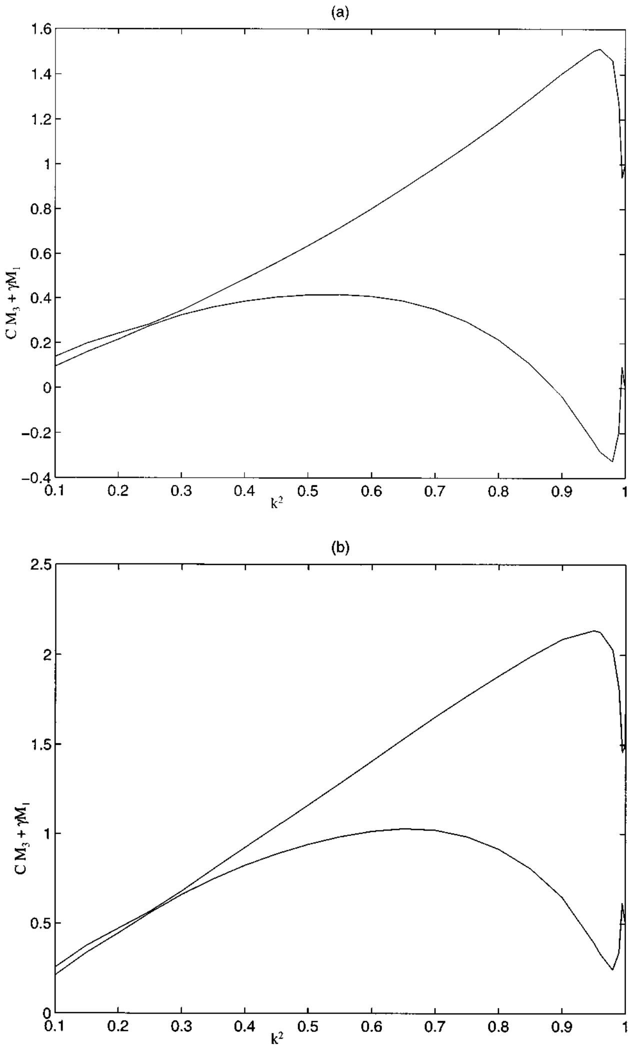

FIG. 6. Locus of the values of the total Melnikov function due to the inertial force and the perturbation in y, $CM_3+\gamma M_1$ as a function of k^2 for a fixed value of $C=0.1$ and two different values for γ, (a) $\gamma=0.05$ and (b) $\gamma=0.1$. It is seen that for large enough values of γ the total Melnikov function can be bounded away from zero. The locus of the values is given by the region bounded by $\gamma M_1+C\min_{\tau\in[0,2\pi]}[M_3(\tau)]$ and $\gamma M_1+C\max_{\tau\in[0,2\pi]}[M_3(\tau)]$.

locus of the values of the Melnikov function $CM_3+\gamma M_1$ as a function of k^2. The parameter C is kept at the constant value $C=0.1$ and γ takes the value $\gamma=0.05$ in Fig. 6(a) and $\gamma=0.1$ in Fig. 6(b), respectively. It is easily seen that if γ is small enough, there will be values of the phase τ for which the Melnikov function will be equal to 0. This will mean the existence of periodic orbits of the same period as the unperturbed one in regions of phase space with the appropriate values of k. However, if γ gets larger, we see that the Melnikov function can never be zero so that no periodic orbits of the same period as the unperturbed ones will survive under the action of the perturbation due to particle inertia.

Finally, it is interesting to consider these Melnikov functions on the separatrix orbit. The Fourier-series expansion is not satisfactory close to the separatrix and so we chose to perform the integrations using the separatrix solution. The Melnikov functions for general values of A and B are

$$M_1=16\lambda^2\left(\frac{-r^2+2r^2\lambda t_0+2\lambda t_0+1}{(r^2-1)^2}\right)r, \qquad (4.24)$$

where $r=\exp(2\lambda t_0)$ and t_0 is given by

$$t_0=\frac{1}{\lambda}\operatorname{arccosh}[(A/B)^{1/2}] \qquad (4.25)$$

and M_3 by

$$M_3=\frac{2\pi}{A-B}\operatorname{sech}\left(\frac{\pi}{2\sigma}\right)(\sin(\nu)-\cos(\nu))\sin(\tau), \quad (4.26)$$

where

$$\nu=\frac{1-m^2}{m}\ln\left(\frac{(1+m)}{(1-m^2)^{1/2}}\right), \quad m^2=\frac{B}{A}, \quad \sigma=\frac{(AB)^{1/2}}{A-B}. \qquad (4.27)$$

For the case $A=2$ and $B=1$ we find, neglecting the Basset contribution,

$$M_t=0.852C\sin(\tau)+\gamma 4.919\,62. \qquad (4.28)$$

We see that this function will never have a zero for any value of τ if $|\gamma|>0.173|C|$. Since simple zeros of the total Melnikov function calculated on the unperturbed separatrix orbit imply the existence of chaotic dynamics in the separatrix region, this result can be interpreted as suppression of chaos due to particle inertia for relatively small values of the parameter γ. However, one must be cautious about this interpretation because the insertion of inertial terms makes the particle dynamics dissipative. The possible suppression of chaos due to particle inertia is in accordance with numerical results of McLaughlin [9]. One should note the relevance of this result in mixing problems: The suppression of chaos in the separatrix region because of the effect of finite size of the advected particles will result in a less efficient mixing of heavy tracers or bubbles than for a passive tracer with the same density as the fluid.

In the above results we have not included the effect of the Basset force. It can easily be seen that the inclusion of the Basset force will not change these qualitative results significantly but will only alter the values of γ for which the suppression of chaos and the destruction of the periodic orbits occurs. It should be noted that one can study the generation of subharmonic orbits in a similar manner by taking an appropriate generalization of the Melnikov functions [10].

V. A SECOND EXAMPLE: SPHERICAL PARTICLES IN THE HILLS VORTEX

As a second example, which produces results similar to the ABC flow, we consider the motion of small spherical particles in the Hill vortex. This is a three-dimensional flow, which in cylindrical coordinates takes the form

$$u_r=rz, \quad u_\theta=\frac{c}{r^2}, \quad u_z=1-2r^2-z^2 \qquad (5.1)$$

and is a solution of the Euler equation everywhere apart from the z axis. This flow is clearly singular since it has a singular

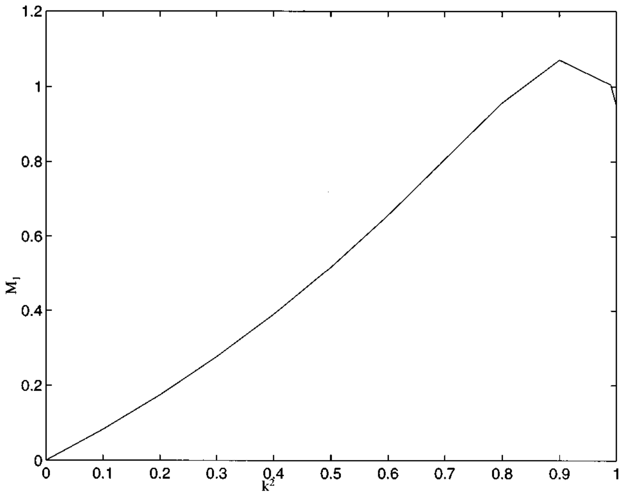

FIG. 7. Inertial force contribution to the Melnikov function for the Hill vortex as a function of k^2 for $c=0$. This Melnikov function is seen to keep its sign for all values of k (that is, throughout the vortex).

vorticity distribution and the u_θ velocity component can become infinite for $r\to 0$. The equations of motion for small spherical particles including only the inertial force is

$$\dot{r}=rz-\gamma\frac{-r^4+2r^6+4c^2}{r^3},\quad \dot{\theta}=\frac{2c}{r^2},$$

$$\dot{z}=1-2r^2-z^2+\gamma(-2z+2z^3). \tag{5.2}$$

We clearly see that the equations for r and z can be separated from the equation for the evolution of θ. We also see that unless $c=0$ the equation for r is singular for small r (near the z axis). We believe that in this case the above model is not a particularly good approximation for the dynamics of heavy particles since the expression of the inertial forces given here is based on a perturbation argument that will break down if the velocity field becomes singular. However, in order to show that the techniques and results given for the ABC flow extend to other cases we give here the Melnikov function for this case bearing in mind that it is strictly valid only if $c=0$ or if the orbits stay sufficiently far from the region $r=0$.

It is useful to define the new variable $R=r^2/2$. The unperturbed system is Hamiltonian in the (R,z) variables and the Hamiltonian is $H=Rz^2-R+2R^2$. The Melnikov function for the effect of the particle inertia on the motion is

$$M_1=-2\gamma\int_{-T/2}^{T/2}\left(2R^2+2c^2-3H+12RH+\frac{c^2H}{R^2}-\frac{2H^2}{R}\right)dt, \tag{5.3}$$

where T is the unperturbed period of the motion in the Hill vortex. We can rewrite this as an integral over R, namely,

$$M=-2\gamma\int_{R_{\min}}^{R_{\max}}\left(2R^2+2c^2-3H+12RH+\frac{c^2H}{R^2}-\frac{2H^2}{R}\right)$$

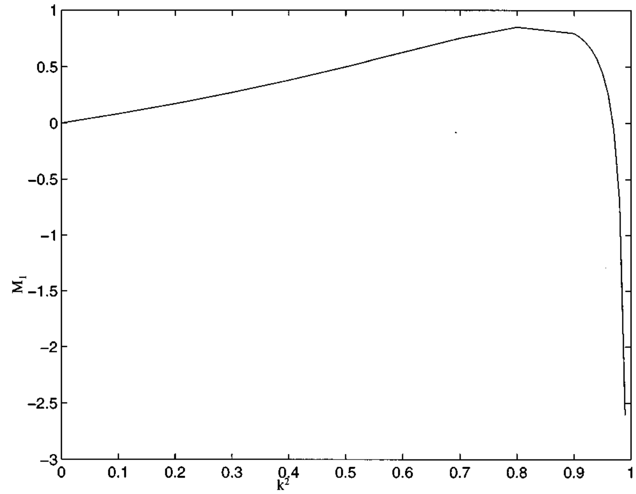

FIG. 8. Inertial force contribution to the Melnikov function for the Hill vortex as a function of k^2 for $c=0.1$. The Melnikov function keeps its sign for values of k far from the separatrix region $k=1$. A zero of the Melnikov function close to the separatrix is associated not with the existence of a limit cycle but with the singularity of the velocity field in this region which makes the validity of the perturbation theory used to derive the equations of motion questionable (see the text for details).

$$\times[R(R_{\max}-R)(R-R_{\min})]^{-1/2}dR, \tag{5.4}$$

where $R_{\max}=1+(1+8H)^{1/2}$ and $R_{\min}=1-(1+8H)^{1/2}$. For more details on the unperturbed motion see [5]. This integral can be expressed analytically in closed form to give

$$M_1=-2\gamma[A(H)K(k)+B(H)E(k)], \tag{5.5}$$

where K and E are the complete elliptic integrals of first and second kind and

$$A(H)=-\frac{4k'^2R_{\max}^{3/2}}{3}-\frac{6H}{R_{\max}^{1/2}}+\frac{4c^2}{R_{\max}^{1/2}}-\frac{2Hc^2}{3R_{\max}^{5/2}k'^2},$$

$$B(H)=\frac{8R_{\max}^{3/2}(1+k'^2)}{3}+24HR_{\max}^{1/2}-\frac{4H^2}{R_{\max}^{3/2}}+\frac{4Hc^2}{3R_{\max}^{5/2}}\frac{(2-k^2)}{k'^4}, \tag{5.6}$$

where $k^2=1-R_{\min}/R_{\max}$ and $k'^2=1-k^2$. For $c=0$ we plot the Melnikov function as a function of k^2 in Fig. 7. We see that this Melnikov function keeps the same sign for all values of H so that particles lighter than the fluid will tend to concentrate in the core of the vortex and heavy particles will be ejected towards the separatrix region. This is similar to the behavior for the ABC flow. For $c\neq 0$ the Melnikov function can change sign as shown in Fig. 8 for $c=0.1$. We can associate this change of sign with the existence of a limit cycle as long as the zero of the Melnikov function is not in the vicinity of the separatrix, where the basic model is no longer valid. (We expect the basic model used here to be valid in the whole phase plane excluding a neighborhood of the separatrix, the width of which clearly depends on the value of c.)

VI. CONCLUSION

In this paper we give some analytical results on the motion of small spherical particles in two distinct incompressible three-dimensional velocity fields in the weak-inertia limit. We give a general justification that for a number of flows the problem can be reduced to the study of a two-dimensional Hamiltonian system with a non-Hamiltonian perturbation and a third degree of freedom whose evolution can be reduced to quadrature. As a result of this some perturbative analytical methods can be used to state results about the bifurcation of limit cycles from the center or the behavior of heteroclinic cycles. As an explicit example we take the motion of inertial particles in the ABC flow, where using these techniques we can prove the nonexistence of limit cycles for small enough values of the parameter γ as well as obtain analytical results on the motion of heavy and light particles in this flow and the relative importance of the inertia and the Basset force contribution throughout the vortex region. An interesting result is the possible suppression of chaos in certain circumstances for bubbles or heavy particles advected by the flow. Since chaos is usually related to effective mixing of a passive tracer advected by a flow field, this result may be interpreted largely as an inhibition of effective mixing of heavy particles or bubbles for certain parameter values. Such results can be of great interest in applications such as environmental sciences or in experimental fluid mechanics where usually tracers have a density different from that of the fluid.

As a second example we consider the motion of particles in the Hill vortex. Similar results can be obtained, namely, that the qualitative behavior of the motion of the tracer can be given by the study of some properly defined Melnikov integral and that for suitable parameter values no limit cycles for the motion of the tracer are observed. As before, bubbles tend to move to the center of the vortex, whereas heavy particles tend to move away from it.

ACKNOWLEDGMENTS

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APPENDIX

In this appendix we very briefly give the multiple-time-scale technique that leads to the result that the Melnikov function is associated with the change of the unperturbed Hamiltonian over a period. We restrict our derivation to the ABC flow, but all the results are applicable to more general flows. Neglecting the Basset force and taking $C=0$, we obtain from the definition of h as given in Sec. IV and the equations of motion given by Eq. (4.3)

$$\dot{h}=\gamma AB[\cos^2(x)\cos(z)+\sin(x)\sin^2(z)]. \tag{A1}$$

We now expand the orbits and write $x=x_0+\gamma x_1$ and $z=z_0+\gamma z_1$, substitute into the differential equation, and expand as a power series in γ. Then, to lowest order

$$\dot{h}=\gamma AB[\cos^2(x_0)\cos(z_0)+\sin(x_0)\sin^2(z_0)], \tag{A2}$$

which can be integrated in time to give the change of the Hamiltonian over an unperturbed period. The integral over the right-hand side of Eq. (A2) then reduces to the Melnikov function. Then introducing a slow time scale $t_1=\gamma t$ we may write

$$\frac{dh}{dt_1}=M_1, \tag{A3}$$

which reflects the fact that h changes only on the slow time scale. Simlarly, we obtain that the Basset force will contribute to this change a factor of $\gamma\epsilon M_2(0)$.

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